

If your problem requires a series of related binomial coefficients, a good idea is to use recurrence relations, for example

$$\binom{n+1}{k} = \frac{n+1}{n-k+1} \binom{n}{k} = \binom{n}{k} + \binom{n}{k-1}$$

$$\binom{n}{k+1} = \frac{n-k}{k+1} \binom{n}{k}$$
(6.1.7)

Finally, turning away from the combinatorial functions with integer valued arguments, we come to the beta function,

$$B(z, w) = B(w, z) = \int_0^1 t^{z-1} (1-t)^{w-1} dt$$
(6.1.8)

which is related to the gamma function by

$$B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$$
(6.1.9)

hence

```

FUNCTION beta(z,w)
REAL beta,w,z
C USES gammaln
  Returns the value of the beta function B(z,w).
REAL gammaln
beta=exp(gammaln(z)+gammaln(w)-gammaln(z+w))
return
END
```

#### CITED REFERENCES AND FURTHER READING:

Abramowitz, M., and Stegun, I.A. 1964, *Handbook of Mathematical Functions*, Applied Mathematics Series, Volume 55 (Washington: National Bureau of Standards; reprinted 1968 by Dover Publications, New York), Chapter 6.

Lanczos, C. 1964, *SIAM Journal on Numerical Analysis*, ser. B, vol. 1, pp. 86–96. [1]

## 6.2 Incomplete Gamma Function, Error Function, Chi-Square Probability Function, Cumulative Poisson Function

The incomplete gamma function is defined by

$$P(a, x) \equiv \frac{\gamma(a, x)}{\Gamma(a)} \equiv \frac{1}{\Gamma(a)} \int_0^x e^{-t} t^{a-1} dt \quad (a > 0)$$
(6.2.1)

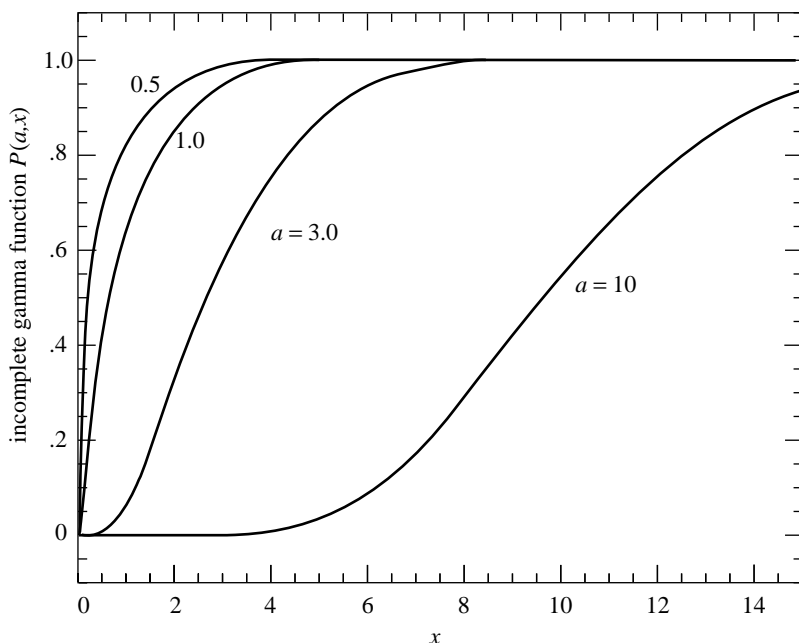


Figure 6.2.1. The incomplete gamma function  $P(a, x)$  for four values of  $a$ .

It has the limiting values

$$P(a, 0) = 0 \quad \text{and} \quad P(a, \infty) = 1 \quad (6.2.2)$$

The incomplete gamma function  $P(a, x)$  is monotonic and (for  $a$  greater than one or so) rises from “near-zero” to “near-unity” in a range of  $x$  centered on about  $a - 1$ , and of width about  $\sqrt{a}$  (see Figure 6.2.1).

The complement of  $P(a, x)$  is also confusingly called an incomplete gamma function,

$$Q(a, x) \equiv 1 - P(a, x) \equiv \frac{\Gamma(a, x)}{\Gamma(a)} \equiv \frac{1}{\Gamma(a)} \int_x^\infty e^{-t} t^{a-1} dt \quad (a > 0) \quad (6.2.3)$$

It has the limiting values

$$Q(a, 0) = 1 \quad \text{and} \quad Q(a, \infty) = 0 \quad (6.2.4)$$

The notations  $P(a, x)$ ,  $\gamma(a, x)$ , and  $\Gamma(a, x)$  are standard; the notation  $Q(a, x)$  is specific to this book.

There is a series development for  $\gamma(a, x)$  as follows:

$$\gamma(a, x) = e^{-x} x^a \sum_{n=0}^{\infty} \frac{\Gamma(a)}{\Gamma(a+1+n)} x^n \quad (6.2.5)$$

One does not actually need to compute a new  $\Gamma(a+1+n)$  for each  $n$ ; one rather uses equation (6.1.3) and the previous coefficient.

A continued fraction development for  $\Gamma(a, x)$  is

$$\Gamma(a, x) = e^{-x} x^a \left( \frac{1}{x+1} \frac{1-a}{1+} \frac{1}{x+} \frac{2-a}{1+} \frac{2}{x+} \dots \right) \quad (x > 0) \quad (6.2.6)$$

It is computationally better to use the even part of (6.2.6), which converges twice as fast (see §5.2):

$$\Gamma(a, x) = e^{-x} x^a \left( \frac{1}{x+1-a-} \frac{1 \cdot (1-a)}{x+3-a-} \frac{2 \cdot (2-a)}{x+5-a-} \dots \right) \quad (x > 0) \quad (6.2.7)$$

It turns out that (6.2.5) converges rapidly for  $x$  less than about  $a + 1$ , while (6.2.6) or (6.2.7) converges rapidly for  $x$  greater than about  $a + 1$ . In these respective regimes each requires at most a few times  $\sqrt{a}$  terms to converge, and this many only near  $x = a$ , where the incomplete gamma functions are varying most rapidly. Thus (6.2.5) and (6.2.7) together allow evaluation of the function for all positive  $a$  and  $x$ . An extra dividend is that we never need compute a function value near zero by subtracting two nearly equal numbers. The higher-level functions that return  $P(a, x)$  and  $Q(a, x)$  are

```

FUNCTION gammp(a,x)
REAL a,gammp,x
C USES gcf,gser
  Returns the incomplete gamma function P(a,x).
REAL gammcf,gamser,gln
if(x.lt.0..or.a.le.0.)pause 'bad arguments in gammp'
if(x.lt.a+1.)then      Use the series representation.
  call gser(gamser,a,x,gln)
  gammp=gamser
else
  Use the continued fraction representation
  call gcf(gammcf,a,x,gln)
  gammp=1.-gammcf      and take its complement.
endif
return
END

```

```

FUNCTION gammq(a,x)
REAL a,gammq,x
C USES gcf,gser
  Returns the incomplete gamma function Q(a,x) ≡ 1 - P(a,x).
REAL gammcf,gamser,gln
if(x.lt.0..or.a.le.0.)pause 'bad arguments in gammq'
if(x.lt.a+1.)then      Use the series representation
  call gser(gamser,a,x,gln)
  gammq=1.-gamser      and take its complement.
else
  Use the continued fraction representation.
  call gcf(gammcf,a,x,gln)
  gammq=gammcf
endif
return
END

```

The argument `gln` is returned by both the series and continued fraction procedures containing the value  $\ln \Gamma(a)$ ; the reason for this is so that it is available to you if you want to modify the above two procedures to give  $\gamma(a, x)$  and  $\Gamma(a, x)$ , in addition to  $P(a, x)$  and  $Q(a, x)$  (cf. equations 6.2.1 and 6.2.3).

The procedures `gser` and `gcf` which implement (6.2.5) and (6.2.7) are

```

SUBROUTINE gser(gamser,a,x,gln)
INTEGER ITMAX
REAL a,gamser,gln,x,EPS
PARAMETER (ITMAX=100,EPS=3.e-7)
C USES gammln
  Returns the incomplete gamma function  $P(a, x)$  evaluated by its series representation as
  gamser. Also returns  $\ln \Gamma(a)$  as gln.
INTEGER n
REAL ap,del,sum,gammln
gln=gammln(a)
if(x.le.0.)then
  if(x.lt.0.)pause 'x < 0 in gser'
  gamser=0.
  return
endif
ap=a
sum=1./a
del=sum
do 11 n=1,ITMAX
  ap=ap+1.
  del=del*x/ap
  sum=sum+del
  if(abs(del).lt.abs(sum)*EPS)goto 1
enddo 11
pause 'a too large, ITMAX too small in gser'
1 gamser=sum*exp(-x+a*log(x)-gln)
return
END

SUBROUTINE gcf(gammcf,a,x,gln)
INTEGER ITMAX
REAL a,gammcf,gln,x,EPS,FPMIN
PARAMETER (ITMAX=100,EPS=3.e-7,FPMIN=1.e-30)
C USES gammln
  Returns the incomplete gamma function  $Q(a, x)$  evaluated by its continued fraction repre-
  sentation as gammcf. Also returns  $\ln \Gamma(a)$  as gln.
  Parameters: ITMAX is the maximum allowed number of iterations; EPS is the relative accu-
  racy; FPMIN is a number near the smallest representable floating-point number.
INTEGER i
REAL an,b,c,d,del,h,gammln
gln=gammln(a)
b=x+1.-a
c=1./FPMIN
d=1./b
h=d
do 11 i=1,ITMAX
  an=-i*(i-a)
  b=b+2.
  d=an*d+b
  if(abs(d).lt.FPMIN)d=FPMIN
  c=b+an/c
  if(abs(c).lt.FPMIN)c=FPMIN
  d=1./d
  del=d*c
  Set up for evaluating continued fraction by modified
  Lentz's method (§5.2) with  $b_0 = 0$ .
  Iterate to convergence.
enddo 11

```

```

      h=h*del
      if(abs(del-1.)<.lt.EPS)goto 1
    enddo !!
    pause 'a too large, ITMAX too small in gcf'
1  gammcf=exp(-x+a*log(x)-gln)*h      Put factors in front.
    return
  END

```

## Error Function

The error function and complementary error function are special cases of the incomplete gamma function, and are obtained moderately efficiently by the above procedures. Their definitions are

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad (6.2.8)$$

and

$$\operatorname{erfc}(x) \equiv 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt \quad (6.2.9)$$

The functions have the following limiting values and symmetries:

$$\operatorname{erf}(0) = 0 \quad \operatorname{erf}(\infty) = 1 \quad \operatorname{erf}(-x) = -\operatorname{erf}(x) \quad (6.2.10)$$

$$\operatorname{erfc}(0) = 1 \quad \operatorname{erfc}(\infty) = 0 \quad \operatorname{erfc}(-x) = 2 - \operatorname{erfc}(x) \quad (6.2.11)$$

They are related to the incomplete gamma functions by

$$\operatorname{erf}(x) = P\left(\frac{1}{2}, x^2\right) \quad (x \geq 0) \quad (6.2.12)$$

and

$$\operatorname{erfc}(x) = Q\left(\frac{1}{2}, x^2\right) \quad (x \geq 0) \quad (6.2.13)$$

Hence we have

```

FUNCTION erf(x)
  REAL erf,x
  C  USES gammp
      Returns the error function erf(x).
  REAL gammp
  if(x<.0)then
    erf=-gammp(.5,x**2)
  else
    erf=gammp(.5,x**2)
  endif
  return
END

```

```

FUNCTION erfc(x)
REAL erfc,x
C USES gammp,gammq
  Returns the complementary error function erfc(x).
REAL gammp,gammq
if(x.lt.0.)then
  erfc=1.+gammp(.5,x**2)
else
  erfc=gammq(.5,x**2)
endif
return
END

```

If you care to do so, you can easily remedy the minor inefficiency in `erf` and `erfc`, namely that  $\Gamma(0.5) = \sqrt{\pi}$  is computed unnecessarily when `gammp` or `gammq` is called. Before you do that, however, you might wish to consider the following routine, based on Chebyshev fitting to an inspired guess as to the functional form:

```

FUNCTION erfcc(x)
REAL erfcc,x
  Returns the complementary error function erfc(x) with fractional error everywhere less than
   $1.2 \times 10^{-7}$ .
REAL t,z
z=abs(x)
t=1./(1.+0.5*z)
erfcc=t*exp(-z*z-1.26551223+t*(1.00002368+t*(.37409196+
*   t*(.09678418+t*(-.18628806+t*(.27886807+t*(-1.13520398+
*   t*(1.48851587+t*(-.82215223+t*.17087277)))))))))
if(x.lt.0.)erfcc=2.-erfcc
return
END

```

There are also some functions of *two* variables that are special cases of the incomplete gamma function:

### Cumulative Poisson Probability Function

$P_x(< k)$ , for positive  $x$  and integer  $k \geq 1$ , denotes the *cumulative Poisson probability* function. It is defined as the probability that the number of Poisson random events occurring will be between 0 and  $k - 1$  *inclusive*, if the expected mean number is  $x$ . It has the limiting values

$$P_x(< 1) = e^{-x} \quad P_x(< \infty) = 1 \quad (6.2.14)$$

Its relation to the incomplete gamma function is simply

$$P_x(< k) = Q(k, x) = \text{gammq}(k, x) \quad (6.2.15)$$

### Chi-Square Probability Function

$P(\chi^2|\nu)$  is defined as the probability that the observed chi-square for a correct model should be less than a value  $\chi^2$ . (We will discuss the use of this function in Chapter 15.) Its complement  $Q(\chi^2|\nu)$  is the probability that the observed chi-square will exceed the value  $\chi^2$  by chance *even* for a correct model. In both cases  $\nu$  is an integer, the number of degrees of freedom. The functions have the limiting values

$$P(0|\nu) = 0 \quad P(\infty|\nu) = 1 \quad (6.2.16)$$

$$Q(0|\nu) = 1 \quad Q(\infty|\nu) = 0 \quad (6.2.17)$$

and the following relation to the incomplete gamma functions,

$$P(\chi^2|\nu) = P\left(\frac{\nu}{2}, \frac{\chi^2}{2}\right) = \text{gammp}\left(\frac{\nu}{2}, \frac{\chi^2}{2}\right) \quad (6.2.18)$$

$$Q(\chi^2|\nu) = Q\left(\frac{\nu}{2}, \frac{\chi^2}{2}\right) = \text{gammq}\left(\frac{\nu}{2}, \frac{\chi^2}{2}\right) \quad (6.2.19)$$

#### CITED REFERENCES AND FURTHER READING:

Abramowitz, M., and Stegun, I.A. 1964, *Handbook of Mathematical Functions*, Applied Mathematics Series, Volume 55 (Washington: National Bureau of Standards; reprinted 1968 by Dover Publications, New York), Chapters 6, 7, and 26.

Pearson, K. (ed.) 1951, *Tables of the Incomplete Gamma Function* (Cambridge: Cambridge University Press).

## 6.3 Exponential Integrals

The standard definition of the exponential integral is

$$E_n(x) = \int_1^\infty \frac{e^{-xt}}{t^n} dt, \quad x > 0, \quad n = 0, 1, \dots \quad (6.3.1)$$

The function defined by the principal value of the integral

$$\text{Ei}(x) = - \int_{-x}^\infty \frac{e^{-t}}{t} dt = \int_{-\infty}^x \frac{e^t}{t} dt, \quad x > 0 \quad (6.3.2)$$

is also called an exponential integral. Note that  $\text{Ei}(-x)$  is related to  $-E_1(x)$  by analytic continuation.

The function  $E_n(x)$  is a special case of the incomplete gamma function

$$E_n(x) = x^{n-1} \Gamma(1-n, x) \quad (6.3.3)$$